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Some applications of symbolic dynamics techniques to toral skew endomorphisms

S Siboni†

Dipartimento di Ingegneria dei Materiali, Facoltà di Ingegneria, Università di Trento, Mesiano 38050 Trento, Italy

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Abstract. A class of dynamical systems of the 2-torus $\mathbb{T}^2$ is considered. These systems have the form of a skew product between the Bernoulli endomorphism $B_p(x) = px \bmod 1$, $p \in \mathbb{Z} \setminus \{-1, 0, 1\}$, defined on the 1-torus $\mathbb{T} \equiv [0, 1)$ and a translation on $\mathbb{T}$ itself. Symbolic dynamics techniques allow one to single out wide classes of observables which show an exponential decay of correlations. For some observables the rate of correlation decay can be explicitly estimated.

0. Introduction and statement of the results

We study here mappings M_ϕ of the 2-torus $\mathbb{T}^2$ defined by

$$M_\phi(x, y) \equiv (px, y + \phi(x)) \bmod 1 \tag{0.1}$$

where $p \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and ϕ is a suitable real-valued function of the 1-torus $\mathbb{T} = (\mathbb{R}/\mathbb{Z}, +)$. $\mathbb{T}$ is endowed with the distance $d_1(x_1, x_2) \equiv \min\{|x_1 - x_2|, 1 - |x_1 - x_2|\}$, $x_1, x_2 \in \mathbb{T}$, and parametrized by the interval $[0, 1)$. We denote by $\mathfrak{B}_1$ the Borel σ-algebra of $(\mathbb{T}, d_1)$ and by μ_1 the normalized Haar measure on $(\mathbb{T}, \mathfrak{B}_1)$. Analogously, we set $\mathbb{T}^2 \equiv \mathbb{T} \times \mathbb{T} = (\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}, +)$, with the metric $d_2((\xi_1, \xi_2), (\eta_1, \eta_2)) \equiv \max_{i=1,2} d_1(\xi_i, \eta_i)$, $(\xi_1, \xi_2), (\eta_1, \eta_2) \in \mathbb{T}^2$, and introduce on the Borel σ-algebra $\mathfrak{B}_2$ of $(\mathbb{T}^2, d_2)$ the corresponding normalized Haar measure. The unit square $[0, 1)^2$ will provide the usual parametrization. Whenever the real function ϕ is μ_1-measurable, (0.1) can be interpreted as the skew product of the circle maps

$$B_p(x) \equiv px \bmod 1 \qquad x \in \mathbb{T} \tag{0.2}$$

and

$$T_x(y) \equiv y + \phi(x) \bmod 1 \qquad y \in \mathbb{T} \tag{0.3}$$

with respect to the invariant product measure $\mu_2 = \mu_1 \times \mu_1$ on $\mathfrak{B}_2 = \mathfrak{B}_1 \times \mathfrak{B}_1$. The triple $(\mathbb{T}, \mu_1, B_p)$ constitutes the so-called dynamical system acting on the base of the skew product and is a well known Bernoulli endomorphism, whereas $(\mathbb{T}, \mu_1, T_x)$ provides a family of toral translations measurably dependent on x and acting on the fibres. Both base and fibres trivially coincide with $\mathbb{T}$ for this class of mappings.

† E-mail address: stefano.siboni@ing.unitn.it

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With the particular choice $p = 2$ and $\phi(x) = \omega + \varepsilon x$, $\varepsilon, \omega \in \mathbb{R}$, maps like (0.1) were originally considered in relation to models of modulated diffusion for Hamiltonian systems subjected to deterministic noise [1, 2]. More precisely, such models took the form

$$\begin{aligned} \alpha' &= 2\alpha \bmod 1 - \tfrac{1}{2} \\ \theta' &= \theta + \omega + \varepsilon\alpha \bmod [0, 2\pi) \\ j' &= j + V(\theta) \end{aligned} \tag{0.4}$$

$j \in \mathbb{R}$ and $\theta \in [0, 2\pi)$ having a physical interpretation as action and angle variables, respectively. $V(\theta)$ was any 2π-periodic analytic function of the angle θ, with zero mean, whereas $\omega \in \mathbb{R}$ was a constant unperturbed frequency and $\varepsilon \in \mathbb{R}$ a perturbation—but actually not necessarily small—coupling parameter. The 'noise variable' α varied on the 1-torus $\mathbb{T}$, whose definition was the same as above, but with the parametrization $[-\frac{1}{2}, \frac{1}{2})$ instead of $[0, 1)$, chosen to obtain a zero-mean perturbation of the frequency. Under the hypothesis of a Diophantine $\omega/2\pi$, (0.4) exhibits interesting properties, common to more sophisticated transport models, like existence and finiteness of a suitably defined diffusion coefficient [1], involving averages on variables α, θ, analytically provable validity of the so-called random phase approximation [2] or numerical evidence for both the central limit property and the invariance principle [3]. Moreover, the time evolution of the action j appears totally 'slaved' to that of (α, θ), whose dynamics is determined by the first two equations in (0.4). These features suggest that the driving dynamical system on the (α, θ) space has rather strong ergodic and statistical properties, but since such a system is trivially conjugated to maps of the kind (0.1), with $p = 2$ and $\phi(x) = \omega + \varepsilon x$, a 'good' ergodic and statistical behaviour of (0.1) itself is expected for this choice of p and ϕ.

In fact, a complete characterization of ergodicity, weak and strong mixing was given in [4] for any value of the parameters ε and ω, whereas the exactness of the skew product for irrational ε was proved in [5] by Parry, who also applied Perron–Frobenius techniques to show that the correlation decay of characters is exponential. More recently, spectral methods have been successfully used to deal with the case of arbitrary $p \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and to estimate the rate of correlation decay for analytic and sufficiently smooth (depending on the choice of p, ε, ω) observables [6]. A very interesting result established in [7] states that the maps $M_\varepsilon(x, y) = (2x, y + \varepsilon x) \bmod 1$ of the 2-torus and B_2 on the circle are isomorphic, whenever the irrational parameter ε is extremely well approximated by rationals (in a suitable sense). Finally, a discussion about the existence and computation of Liapunov exponents and Kolmogorov–Simai-entropy of (0.1), under quite general assumptions about ϕ, can be found in [8]. It is clear that, according to the choice of ϕ, the map may present very different kinds of ergodic behaviour, from Bernoullicity to ergodicity, weak and strong mixing, exactness or even lack of ergodicity. On the one hand, the occurrence of ergodicity can be fully characterized by using Anzai's criterion [9, 10] and many examples of non-ergodic endomorphisms M_ϕ easily provided. On the other hand, it is known that when $\phi(x) = \omega + \varepsilon x$ exactness of M_ϕ occurs if and only if $\varepsilon \in \mathbb{R} \setminus \mathbb{Q}$ [5], whereas $\varepsilon \in \mathbb{Q}$ implies the map to be ergodic or non-ergodic according to the irrationality or rationality of ω, respectively [4].

In this work we tackle two different kinds of problems. First of all we generalize the result given in [5] and show the exponential decay of correlations for characters under quite weak assumptions on ϕ in (0.1)—with respect to the (natural) invariant measure μ_2. By character we mean [11] any (continuous) homomorphism of an Abelian group into the multiplicative group of the complex numbers of modulus 1: throughout the paper the Abelian group we will consider is the additive 2-torus $\mathbb{T}^2 = (\mathbb{R}^2/\mathbb{Z}^2, +)$.

The proof of the following proposition is deferred to section 1.

Proposition 1. Let ϕ be Lipschitz continuous in $\mathbb{T}$ and let the toral endomorphism M_ϕ defined by (0.1) be weak mixing with respect to the invariant measure μ_2. Then the correlations of characters decay exponentially.

This statement entails, in particular, the equivalence of weak and strong mixing. Indeed, endomorphisms like (0.1) have a natural inverse limit as invertible systems which are skew extensions of a *two*-sided Bernoulli shift. Weak and strong mixing of an endomorphism and its inverse limit are the same. In this case no difference between weak and strong mixing occurs, as both are equivalent to the invertible extension being Bernoulli [12].

From a physical point of view, the most significant by-product of the previous result is that, for this kind of maps, (weak and strong) mixing occurs if and only if the correlations of any two elements of the Fourier basis on $\mathbb{T}^2$ decay exponentially. In particular, even if only a unique pair of Fourier vectors exists whose correlations do not decay at an exponential rate, the map can be at most ergodic.

The second problem we address concerns more specifically the case $\phi(x) = \omega + \varepsilon x$ and, in particular, the rate of correlation decay when the mixing condition $\varepsilon \in \mathbb{R} \setminus \mathbb{Q}$ is satisfied. In [5] the exponential decay of correlations for characters is proved, but with no explicit estimate of the decay rate. Such an exponential rate is exactly reckoned in [6] for the correlations of any character; there, this result allows one to use spectral techniques and provides upper bounds to the correlations of smooth and analytic observables. The correlation decay turns out to be faster than any power $n^{-\gamma}$, $\gamma > 0$, in the analytic case, whereas smooth observables satisfy a power-law decay. Nonetheless, these weaker than exponential bounds may be far from being optimal, especially if we consider that particular examples of *discontinuous* observables obeying an exponential decay law can be found even in the purely ergodic case ($\varepsilon \in \mathbb{Q}$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$). Furthermore, a somewhat unnatural Diophantine condition on ε is needed in the proofs and the 'good' symbolic dynamics of the Bernoulli system B_p acting on the base seems to play no evident role.

Proposition 2 singles out a wide class of observables whose correlations decay exponentially, with no supplementary requirement on ε other than irrationality. As will become clear in section 2, the application of the symbolic dynamics defined for B_p is crucial to achieve the result, together with a rather cumbersome estimate established in [13] for different purposes. Finally, a simple comment on the proof leads to the explicit construction of a family of discontinuous observables also showing an exponential decay of correlations.

Proposition 2. Let M be any mixing endomorphism of $\mathbb{T}^2$ defined by

$$M(x, y) \equiv (px, y + \omega + \varepsilon x) \bmod 1 \tag{0.5}$$

with $p \in \mathbb{Z} \setminus \{-1, 0, 1\}$, $\varepsilon \in \mathbb{R} \setminus \mathbb{Q}$ and $\omega \in \mathbb{R}$. Consider a zero-mean observable of the form

$$f(z) = g(x)\,\mathrm{e}^{\mathrm{i}2\pi by} \qquad \forall z \equiv (x, y) \in \mathbb{T}^2 \tag{0.6}$$

where $b \in \mathbb{Z} \setminus 0$ and $g : \mathbb{T} \to \mathbb{R}$ is a Hölder continuous function with exponent $\alpha \in (0, 1]$. Then the autocorrelations of f decay at least exponentially

$$\left| \int_{\mathbb{T}^2} \overline{f(M^n(z))}\, f(z)\, \mathrm{d}\mu_2(z) \right| \leqslant K\theta^n \qquad \forall n \in \mathbb{Z}_+$$

with a suitable constant $K > 0$ and rate

$$\theta = \left\{ \frac{\alpha \ln|p| \ln|J(p, \varepsilon b)|}{\alpha \ln|p| - \ln|J(p, \varepsilon b)|} \right\} < 1 \tag{0.7}$$

on having introduced the notation

$$J(p, \varepsilon b) \equiv \sin\left(p\frac{\pi\varepsilon b}{p-1}\right)\left[p\sin\left(\frac{\pi\varepsilon b}{p-1}\right)\right]^{-1}.$$

The condition $b \neq 0$ is only imposed to avoid trivial cases; the occurrence of exponential correlation decay for Hölder continuous observables dependent on the only variable x is simply ensured by the Bernoullicity of B_p. A slight modification of the proof allows us to obtain an analogous result for the purely ergodic case $\varepsilon \in \mathbb{Q}$, provided that a suitable 'non-resonance' condition is posed on b (see later).

An immediate corollary of proposition 2 is that any observable consisting of a finite linear combination of functions like (0.6) also decays exponentially, and at a computable rate. Of course, countable linear combinations of (0.6) can be analysed by spectral methods, leading to estimates similar to those established in [6].

1. Proof of proposition 1

The proof is largely based on the arguments developed in [5]. Let ϕ be Lipschitz continuous in $\mathbb{T}$ and consider any non-constant character of $\mathbb{T}^2$, that is

$$e_{a,b}(x, y) \equiv \mathrm{e}^{\mathrm{i}2\pi(ax+by)} \qquad (a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\}, (x, y) \in \mathbb{T}^2. \tag{1.1}$$

We preliminary observe that $\forall n \geqslant 1$ the definition (0.1) implies

$$M_\phi^n(x_0, y_0) \equiv (x_n, y_n) = \left(p^n x_0, y_0 + \sum_{j=0}^{n-1}\phi(x_j)\right) \bmod 1 \tag{1.2}$$

so that the autocorrelations of $e_{a,b}$

$$C_n(a, b) \equiv \int_{\mathbb{T}^2} \overline{e_{a,b}(x, y)} e_{a,b} \circ M_\phi^n(x, y)\,\mathrm{d}\mu_2(x, y) \qquad n \in \mathbb{N} \tag{1.3}$$

become

$$C_n(a, b) = \int_{\mathbb{T}} \exp\left\{\mathrm{i}2\pi\left[a(p^n-1)x_0 + b\sum_{j=0}^{n-1}\phi(x_j)\right]\right\}\mathrm{d}\mu_1(x_0) \tag{1.4}$$

and since $a(p^n-1)x_0 \bmod 1 = a(p-1)\sum_{j=0}^{n-1} x_j \bmod 1$, take the equivalent form

$$\begin{aligned} C_n(a, b) &= \int_{\mathbb{T}} \exp\left\{\mathrm{i}2\pi\sum_{j=0}^{n-1}[a(p-1)x_j + b\phi(x_j)]\right\}\mathrm{d}\mu_1(x_0) \\ &= \int_{\mathbb{T}} \exp\left\{\mathrm{i}2\pi\sum_{j=0}^{n-1}\lambda(x_j)\right\}\mathrm{d}\mu_1(x_0) \end{aligned} \tag{1.5}$$

with $\lambda(x) \equiv a(p-1)x + b\phi(x)$. From now on we will consider, for simplicity's sake, the case of positive p, but it is understood that the final results are still valid when $p \leqslant -2$, even if the calculations are slightly different. Following [5] we introduce the symbolic dynamics of the Bernoulli map B_p. In the space $\Sigma_p^+ \equiv \prod_{i=0}^{\infty}\{0, 1, \ldots, p-1\}$ of the one-sided sequences of symbols $0, 1, 2, \ldots, p-1$, let $\mathfrak{M}_p$ be the σ-algebra generated by the 'cylindrical' sets

$$C(\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_{j+l}) \equiv \{\omega \equiv (\omega_i)_{i=0}^{\infty} \in \Sigma_p^+ : \omega_k = \alpha_k\ \forall k = j+1, \ldots j+l\}. \tag{1.6}$$

A probability measure m_p is uniquely determined on $\mathfrak{M}_p$ by posing, for each cylinder, $m_p(C(\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_{j+l})) \equiv p^{-l}$, and the shift map $\sigma(\omega) = \omega'$, $\omega'_i = \omega_{i+1}\ \forall i \geqslant 0$, on the probability space $(\Sigma_p^+, \mathfrak{M}_p, m_p)$ admits m_p as an invariant measure. The resulting Bernoulli shift can be easily conjugated (mod 0) with $(B_p, \mathbb{T}, \mathfrak{B}_1, \mu_1)$ by means of the mapping $\chi(\omega) \equiv \sum_{i=0}^{\infty} \omega_i p^{-(i+1)}$, and if D_p stands for the set of p-adic numbers in the unit interval $[0, 1)$, with measure $\mu_1(D_p) = 0$, χ is also one-to-one in $\mathbb{T} \setminus D_p$. It finally maps the Bernoulli measure m_p on Σ_p^+ to the Haar–Lebesgue measure μ_1, so that the autocorrelations (1.5) can be rewritten in the form

$$C_n(a, b) = \int_{\Sigma_p^+} \exp\left\{ \mathrm{i}2\pi \sum_{j=0}^{n-1} \lambda[B_p^j(\chi(\omega))] \right\} \mathrm{d}m_p(\omega) = \int_{\Sigma_p^+} (\mathcal{P}^n 1)(\omega)\, \mathrm{d}m_p(\omega) \tag{1.7}$$

where $1 : \Sigma_p^+ \to \mathbb{R}$ is the constant function of value 1 and $\mathcal{P}$ the complex Ruelle–Perron–Frobenius (RPF) operator

$$(\mathcal{P}h)(\omega) \equiv \sum_{\bar{\omega}:\sigma(\bar{\omega})=\omega} \exp\{-\log p + \mathrm{i}2\pi\lambda[\chi(\bar{\omega})]\} h(\bar{\omega})$$
$$h : \Sigma_p^+ \to \mathbb{C}, h \in L^2(\Sigma_p^+, \mathfrak{M}_p, m_p). \tag{1.8}$$

Note that:

(i) with respect to the distance $d_{1/p} : \Sigma_p^+ \times \Sigma_p^+ \to \mathbb{R}_+$ defined by $d_{1/p}(\omega, \bar{\omega}) \equiv p^{-N}$, N being the largest integer such that $\omega_i = \bar{\omega}_i$, $0 \leqslant i < N$, the conjugation χ is Lipschitz continuous, $|\chi(\omega) - \chi(\bar{\omega})| \leqslant d_{1/p}(\omega, \bar{\omega})$, $\forall \omega, \bar{\omega} \in \Sigma_p^+$, and so is $\lambda \circ \chi$

$$|\lambda \circ \chi(\omega) - \lambda \circ \chi(\bar{\omega})| \leqslant (|a(p-1)| + |b|\kappa) d_{1/p}(\omega, \bar{\omega}) \tag{1.9}$$

where κ is the Lipschitz constant of ϕ;

(ii) the shift (Σ_p^+, σ) is aperiodic [14];

(iii) the associated real RPF operator

$$(\mathcal{P}_r h)(\omega) = \sum_{\bar{\omega}:\sigma(\bar{\omega})=\omega} \mathrm{e}^{-\log p} h(\bar{\omega}) \qquad \forall h : \Sigma_p^+ \to \mathbb{C},\ h \in L^2(\Sigma_p^+, \mathfrak{M}_p, m_p) \tag{1.10}$$

satisfies the normalization property $\mathcal{P}_r 1 = 1$. Therefore, m_p is the only equilibrium probability measure for $\mathcal{P}_r$, according to RPF theorem [15, 16].

The above items ensure (see [16], pp 49–53, and proposition 4.4 in particular) that the RPF operator (1.8), considered on the (suitably normed) space $C(\Sigma_p^+)$ of continuous functions in $(\Sigma_p^+, d_{1/p})$, has a spectral radius strictly less than 1 if the isometric operator

$$(Vh)(\omega) \equiv \mathrm{e}^{-\mathrm{i}2\pi\lambda[\chi(\omega)]} h(\sigma(\omega)) \qquad h : \Sigma_p^+ \to \mathbb{C},\ h \in L^2(\Sigma_p^+, \mathfrak{M}_p, m_p) \tag{1.11}$$

admits no eigenfunction (with eigenvalue necessarily of the form $\mathrm{e}^{\mathrm{i}\alpha}$, a phase factor). The latter condition, reprojected to $L^2(\mathbb{T}, \mathfrak{B}_1, \mu_1)$ via χ, is obviously equivalent to exclude the existence of a constant $\alpha \in \mathbb{R}$ and a μ_1-almost everywhere non-vanishing function $R : \mathbb{T} \to \mathbb{C}$, $R \in L^2(\mathbb{T}, \mathfrak{B}_1, \mu_1)$, such that

$$R(B_p(x)) = \mathrm{e}^{\mathrm{i}2\pi[a(p-1)x + b\phi(x)]} R(x)\, \mathrm{e}^{\mathrm{i}\alpha} \tag{1.12}$$

for μ_1-almost all $x \in \mathbb{T}$. Such a condition cannot occur and we prove this by *reductio ad absurdum*. Thus, suppose (1.12) is verified for some α and R as above. This would imply

$$R(B_p(x))\, \mathrm{e}^{-\mathrm{i}2\pi apx} = \mathrm{e}^{\mathrm{i}2\pi b\phi(x)} R(x)\, \mathrm{e}^{-\mathrm{i}2\pi ax}\, \mathrm{e}^{\mathrm{i}\alpha}$$

and, therefore, by posing $S(x) \equiv \mathrm{e}^{-\mathrm{i}2\pi ax} R(x)$, we would have $S(B_p(x)) = \mathrm{e}^{\mathrm{i}2\pi b\phi(x)} S(x)\, \mathrm{e}^{\mathrm{i}\alpha}$. The $L^2(\mathbb{T}^2, \mathfrak{B}_2, \mu_2)$ function $\psi(x, y) \equiv S(x)\, \mathrm{e}^{-\mathrm{i}2\pi by}$, $(x, y) \in \mathbb{T}^2$, would then satisfy

$$(\psi \circ M_\phi)(x, y) = S(B_p(x))\, \mathrm{e}^{-\mathrm{i}2\pi b(y+\phi(x))} = \mathrm{e}^{\mathrm{i}\alpha}\, S(x)\, \mathrm{e}^{-\mathrm{i}2\pi by} = \mathrm{e}^{\mathrm{i}\alpha} \psi(x, y)$$

and finally $(\psi \circ M_\phi) = \mathrm{e}^{\mathrm{i}\alpha}\psi$ μ_2-almost everywhere on $\mathbb{T}^2$. Notice that whenever $b \neq 0$ the function ψ is a zero-mean observable

$$\int_{\mathbb{T}^2} \psi \, \mathrm{d}\mu_2 = \int_{\mathbb{T}^2} S(x)\, \mathrm{e}^{-\mathrm{i}2\pi b y}\, \mathrm{d}\mu_2(x, y) = \int_{\mathbb{T}} S(x)\, \mathrm{d}\mu_1(x) \int_{\mathbb{T}} \mathrm{e}^{-\mathrm{i}2\pi b y}\, \mathrm{d}\mu_1(y) = 0.$$

For every $n \in \mathbb{N}$ the autocorrelations of ψ are then written as

$$C_n(\psi) = \int_{\mathbb{T}^2} \overline{\psi \circ M_\phi^n}\psi \, \mathrm{d}\mu_2 = \mathrm{e}^{-\mathrm{i}n\alpha} \int_{\mathbb{T}^2} |\psi|^2 \, \mathrm{d}\mu_2$$

but if M_ϕ satisfies a weak mixing property the observable ψ must obey [17]

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} |C_n(\psi)| = 0$$

whereas

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} |C_n(\psi)| = \int_{\mathbb{T}^2} |\psi|^2 \, \mathrm{d}\mu_2$$

which is positive unless $\psi = 0$ μ_2-almost everywhere on $\mathbb{T}^2$. Since $\psi(x, y) = S(x)\, \mathrm{e}^{-\mathrm{i}2\pi b y}$, with $S \in L^2(\mathbb{T}, \mathfrak{B}_1, \mu_1)$, the last statement would imply $S = 0$ μ_1-almost everywhere on $\mathbb{T}$, a contradiction.

From the obvious observation $1 \in C(\Sigma_p^+)$ we conclude that $\forall (a, b) \in \mathbb{Z}^2$, $b \neq 0$, there holds

$$|C_n(a, b)| = \left| \int_{\Sigma_p^+} (\mathcal{P}^n 1)(\omega)\, \mathrm{d}m_p(\omega) \right| \leqslant K r^n \qquad \forall n \in \mathbb{N} \tag{1.13}$$

for some constants $K > 0$ and $r \in (0, 1)$—dependent on (a, b).

As for the case $b = 0$, the exponential decay of autocorrelations for characters trivially follows by noting that the only relevant dynamics is that of the Bernoulli map B_p on the circle—$e_{a,0}$ reduces to a Lipschitz continuous observable on $\mathbb{T}$. We conclude that correlations decay exponentially for all the $L^2(\mathbb{T}^2, \mathfrak{B}_2, \mu_2)$-complete orthonormal set of characters. □

2. Proof of proposition 2

The idea of the proof is very simple. For this kind of map no general argument ensures the existence of a 'dynamical partition' of $\mathbb{T}^2$, which would make it possible to use symbolic dynamics techniques directly on the whole space; this means that no obvious partition of the 2-torus exists whose indicators can approximate *any* observable of $\mathbb{T}^2$ and have correlations decaying at a known, uniform rate. In particular, (0.5) is clearly non-hyperbolic and we do not expect that a Markov partition can be introduced. Nonetheless, such a Markov partition is trivially defined for the Bernoulli endomorphism B_p *on the 1-torus*, and any function $g(x)$ in (0.6) can be conveniently approximated by indicators of cylindrical sets of B_p. The problem is then reduced to providing uniform estimates to the rates of correlation decay for observables of the form (0.6), *g being the indicator of any cylinder of B_p*. This is exactly what it is possible to do, in this case, by using estimates already available in the literature.

In what follows we will denote by $C_n(f)$ the autocorrelations at time $n \in \mathbb{N}$ of the observable f. For simplicity's sake, but with no loss of generality, we confine ourselves to the case of positive p, the proof for $p \leqslant -2$ being completely analogous.

Suppose that g is Hölder continuous with exponent $\alpha \in (0, 1]$ and constant $L > 0$, so that

$$|g(x_1) - g(x_2)| \leqslant L d_1(x_1, x_2)^\alpha \qquad \forall x_1, x_2 \in \mathbb{T}. \tag{2.1}$$

The first step of the proof is a standard application of symbolic dynamics to the estimate of correlation decay. We replace the observable by an approximated observable piecewise constant on the elements of a suitable partition of phase space having good mixing properties through the dynamics. In this case such a partition on the whole probability space $\mathbb{T}^2$ is not trivially available, but the particular structure of f allows us to apply a piecewise constant approximation on the factor $g(x)$ only, by using cylinders of B_p in the base $\mathbb{T}$. Remembering that for any given $m \in \mathbb{N}$ a set of Markov cylinders of B_p consists of the p-adic intervals $(hp^{-m}, (h+1)p^{-m})$, $h = 0, 1, \ldots, p^m - 1$, let us introduce the approximated observable

$$f_m(z) \equiv \mathrm{e}^{\mathrm{i}2\pi b y} \sum_{h=0}^{p^m-1} g(\xi_h) \mathcal{X}_{[h/p^m,(h+1)/p^m)}(x) \tag{2.2}$$

where $\mathcal{X}_\Omega$ denotes the characteristic function of any set $\Omega \subseteq \mathbb{T}$ and $\xi_h \in (hp^{-m}, (h+1)p^{-m})$. We use (2.2) to approximate the autocorrelation $C_n(f)$, $n \in \mathbb{Z}_+$. To this end, we consider the equality

$$\begin{aligned} C_n(f) = \int_{\mathbb{T}^2} \overline{f(M^n(z))} f(z)\, \mathrm{d}\mu_2(z) &= \int_{\mathbb{T}^2} \overline{[f(M^n(z)) - f_m(M^n(z))]} f(z)\, \mathrm{d}\mu_2(z) \\ &+ \int_{\mathbb{T}^2} \overline{f_m(M^n(z))}[f(z) - f_m(z)]\, \mathrm{d}\mu_2(z) + \int_{\mathbb{T}^2} \overline{f_m(M^n(z))} f_m(z)\, \mathrm{d}\mu_2(z) \end{aligned} \tag{2.3}$$

and provide estimates for each integral on the right-hand side. Since f is continuous on the compact $\mathbb{T}^2$, there exists $\|f\|_\infty \equiv \sup_{z\in\mathbb{T}^2} |f(z)|$ and therefore

$$\left| \int_{\mathbb{T}^2} \overline{[f(M^n(z)) - f_m(M^n(z))]} f(z)\, \mathrm{d}\mu_2(z) \right| \leqslant \|f\|_\infty \int_{\mathbb{T}^2} |f(M^n(z)) - f_m(M^n(z))|\, \mathrm{d}\mu_2(z).$$

This upper bound can also be rewritten into the form

$$\begin{aligned} &\|f\|_\infty \int_{\mathbb{T}} \left| g(p^n x \bmod 1) - \sum_{h=0}^{p^m-1} g(\xi_h) \mathcal{X}_{[h/p^m,(h+1)/p^m)}(p^n x \bmod 1) \right| \mathrm{d}\mu_1(x) \\ &\qquad \leqslant \|f\|_\infty \int_{\mathbb{T}} \sum_{h=0}^{p^m-1} |g(p^n x \bmod 1) - g(\xi_h)| \mathcal{X}_{[h/p^m,(h+1)/p^m)}(p^n x \bmod 1)\, \mathrm{d}\mu_1(x) \end{aligned} \tag{2.4}$$

and since for each term in the sum the only values of x to be considered are those satisfying $p^n x \bmod 1 \in [h/p^m, (h+1)/p^m)$, there holds

$$|g(p^n x \bmod 1) - g(\xi_h)| \leqslant L |p^n x \bmod 1 - \xi_h|^\alpha \leqslant L p^{-\alpha m} \qquad \forall h = 0, 1, \ldots, p^m - 1$$

so that (2.4) provides the further bound

$$\|f\|_\infty \int_{\mathbb{T}} \sum_{h=0}^{p^m-1} L p^{-\alpha m} \mathcal{X}_{[h/p^m,(h+1)/p^m)}(p^n x \bmod 1)\, \mathrm{d}\mu_1(x) = \|f\|_\infty L p^{-\alpha m}. \tag{2.5}$$

The second integral on the right-hand side of (2.3) is estimated immediately by observing that $|f_m(M^n(z))| \leqslant \|f_m \circ M^n\|_\infty = \|f_m\|_\infty \leqslant \|f\|_\infty$:

$$\left| \int_{\mathbb{T}^2} \overline{f_m(M^n(z))}[f(z) - f_m(z)]\, \mathrm{d}\mu_2(z) \right| \leqslant \|f\|_\infty \int_{\mathbb{T}^2} |f(z) - f_m(z)|\, \mathrm{d}\mu_2(z) \leqslant \|f\|_\infty L p^{-\alpha m}$$

where the residual integral is bounded as in (2.5). As for the last integral in (2.3), we have the identity

$$\int_{\mathbb{T}^2} \overline{f_m(M^n(z))}\, f_m(z)\,\mathrm{d}\mu_2(z) = \int_{\mathbb{T}^2} \mathrm{e}^{-\mathrm{i}2\pi b y_n} \sum_{h=0}^{p^m-1} g(\xi_h)\mathcal{X}_{[h/p^m,(h+1)/p^m)}(x_n)\,\mathrm{e}^{\mathrm{i}2\pi b y_0}$$
$$\times \sum_{h'=0}^{p^m-1} g(\xi_{h'})\mathcal{X}_{[h'/p^m,(h'+1)/p^m)}(x_0)\,\mathrm{d}\mu_2(x_0, y_0) \qquad (2.6)$$

with $x_n = p^n x_0 \bmod 1$ and $y_n = y_0 + n\omega + \varepsilon \sum_{j=0}^{n-1} x_j$, so that

$$\mathrm{e}^{\mathrm{i}2\pi b(y_0-y_n)} = \exp\left\{\mathrm{i}2\pi b\left(-n\omega - \varepsilon\sum_{j=0}^{n-1} x_j\right)\right\}$$

and therefore (2.6) can be put into the equivalent form

$$\mathrm{e}^{-\mathrm{i}2\pi b\omega n} \sum_{h,h'=0}^{p^m-1} g(\xi_h)g(\xi_{h'}) \int_{\mathbb{T}} \exp\left\{-\mathrm{i}2\pi b\varepsilon \sum_{j=0}^{n-1} x_j\right\}\mathcal{X}_{[h/p^m,(h+1)/p^m)}(x_n)$$
$$\times\mathcal{X}_{[h'/p^m,(h'+1)/p^m)}(x_0)\,\mathrm{d}\mu_1(x_0).$$

By introducing the notation†

$$I(p,\varepsilon b;k) \equiv \sin\left[p\frac{\pi\varepsilon b}{p-1}(1-p^{-k})\right]\left[p\sin\left[\frac{\pi\varepsilon b}{p-1}(1-p^{-k})\right]\right]^{-1} \qquad \forall k\in\mathbb{N}$$

a simple but rather tedious computation [13] allows us to prove $\forall n, m \in \mathbb{N}$, $n > m$, and $\forall h, h' \in \{0, 1, \ldots, p^m - 1\}$ the basic equality

$$\int_{\mathbb{T}} \exp\left\{-\mathrm{i}2\pi b\varepsilon\sum_{j=0}^{n-1} x_j\right\}\mathcal{X}_{[h/p^m,(h+1)/p^m)}(x_n)\mathcal{X}_{[h'/p^m,(h'+1)/p^m)}(x_0)\,\mathrm{d}\mu_1(x_0)$$
$$= \frac{1}{p^{2m}}\mathrm{sinc}\left[\pi b\varepsilon\frac{p^n-1}{p-1}\frac{1}{p^{m+n}}\right]\prod_{k=m+1}^{n} I(p,\varepsilon b;k)\,\exp\left\{-\mathrm{i}2\pi b\left[\frac{\varepsilon}{2}(n-m)\right.\right.$$
$$\left.\left.+\frac{\varepsilon}{p-1}\left[\Delta_m(h') - \frac{h'}{p^m} + \left(\frac{1}{p^m} - \frac{1}{p^{m+n}}\right)h + \frac{1}{2p^n}\left(1-\frac{1}{p^m}\right)\right]\right]\right\}$$

where $\mathrm{sinc}(z) \equiv \sin z/z\ \forall z \in \mathbb{R}\setminus\{0\}$, $\mathrm{sinc}(0) = 1$ and $\Delta_m(h) \equiv \sum_{k=0}^m a_k$, if $\sum_{k=0}^m a_k\, p^{m-k}$ is the Hindu–Arabic representation in the base p of the integer h, with $a_0 = 0$. The result can easily be found by induction and entails $\forall n, m \in \mathbb{N}$, $n > m$, the upper bound

$$\left|\int_{\mathbb{T}} \exp\left\{-\mathrm{i}2\pi b\varepsilon\sum_{j=0}^{n-1} x_j\right\}\mathcal{X}_{[h/p^m,(h+1)/p^m)}(x_n)\mathcal{X}_{[h'/p^m,(h'+1)/p^m)}(x_0)\,\mathrm{d}\mu_1(x_0)\right|$$
$$\leqslant \frac{1}{p^{2m}}\left|\prod_{k=m+1}^{n} I(p,\varepsilon b;k)\right|$$

uniform on $h, h' \in \{0, 1, \ldots, p^m - 1\}$. Following the main ideas in [13], we use the previous inequality in (2.6) and get

$$\left|\int_{\mathbb{T}^2} \overline{f_m(M^n(z))}\, f_m(z)\,\mathrm{d}\mu_2(z)\right| \leqslant \|f\|_\infty^2 \left|\prod_{k=m+1}^{n} I(p,\varepsilon b;k)\right|.$$

† The function $\sin(px)/px$ is regarded as defined by continuity at $x = 0$.

As a conclusion, $\forall b \in \mathbb{Z}\backslash\{0\}$, $m, n \in \mathbb{N}$, $n > m$, the equality $\int_{\mathbb{T}} f(z)\, \mathrm{d}\mu_2(z) = 0$ holds and

$$|C_n(f)| \leqslant 2L\|f\|_\infty p^{-\alpha m} + \|f\|_\infty^2 \left|\prod_{k=m+1}^{n} I(p, \varepsilon b; k)\right|. \tag{2.7}$$

It is now enough to pose $m \equiv \lfloor rn \rfloor$, with $r \in (0, 1)$ fixed and $\lfloor x \rfloor$ the integer part of $x \in \mathbb{R}$, to deduce the exponential decay of correlations. Note that the limit

$$\lim_{n\to+\infty} \prod_{k=1}^{n} [I(p, \varepsilon b; k) J(p, \varepsilon b)^{-1}]$$

exists and is finite and different from zero owing to the finite upper bound

$$\sum_{k=1}^{n} |I(p, \varepsilon b; k) J(p, \varepsilon b)^{-1} - 1| \leqslant |J(p, \varepsilon b)|^{-1} \sup_{|x|\leqslant\pi} \left|\frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin(px)}{p \sin x}\right| \frac{\pi|\varepsilon b|}{(p-1)^2} \qquad \forall n \in \mathbb{Z}_+$$

and, therefore, two positive constants $\Lambda_+(p, \varepsilon b)$ and $\Lambda_-(p, \varepsilon b)$ can be found such that

$$\Lambda_-(p, \varepsilon b)|J(p, \varepsilon b)|^n < \left|\prod_{k=1}^{n} I(p, \varepsilon b; k)\right| < \Lambda_+(p, \varepsilon b)|J(p, \varepsilon b)|^n \qquad \forall n \in \mathbb{Z}_+.$$

We can then write

$$\left|\prod_{k=m+1}^{n} I(p, \varepsilon b; k)\right| \leqslant \frac{\Lambda_+(p, \varepsilon b)}{\Lambda_-(p, \varepsilon b)} |J(p, \varepsilon b)|^{n-m}.$$

By replacing this inequality inside the upper bound (2.7) we finally obtain

$$|C_n(f)| \leqslant 2L\|f\|_\infty p^{-\alpha m} + \|f\|_\infty^2 \frac{\Lambda_+(p, \varepsilon b)}{\Lambda_-(p, \varepsilon b)} |J(p, \varepsilon b)|^{n-m} \qquad \forall n > m \in \mathbb{N} \tag{2.8}$$

and therefore

$$|C_n(f)| \leqslant 2L\|f\|_\infty p^{-\alpha\lfloor rn \rfloor} + \|f\|_\infty^2 \frac{\Lambda_+(p, \varepsilon b)}{\Lambda_-(p, \varepsilon b)} |J(p, \varepsilon b)|^{n-\lfloor rn \rfloor} \tag{2.9}$$

for every $n \in \mathbb{Z}_+$ and $r \in (0, 1)$ fixed. We only have to choose the parameter r in order to optimize the estimate (2.9). Such a bound is of the type $|C_n(f)| \leqslant A\mu^{\lfloor rn \rfloor} + B\nu^{n-\lfloor rn \rfloor}$ $\forall n \in \mathbb{Z}_+$, with $A, B > 0$ and $\mu, \nu, r \in (0, 1)$, and it can firstly be weakened as follows:

$$|C_n(f)| \leqslant A\, \mathrm{e}^{(rn-1)\ln\mu} + B\, \mathrm{e}^{(n-rn)\ln\nu} = \frac{A}{\mu} \mathrm{e}^{nr\ln\mu} + B\, \mathrm{e}^{n(1-r)\ln\nu}.$$

To achieve optimality, we require that the decay rate of both terms is the same and get

$$|C_n(f)| \leqslant \left(\frac{A}{\mu} + B\right) \exp\left\{\frac{\ln\mu \ln\nu}{\ln\mu + \ln\nu} n\right\}. \tag{2.10}$$

In our case, $A \equiv 2L\|f\|_\infty$, $B \equiv \|f\|_\infty^2 \Lambda_+(p, \varepsilon b)/\Lambda_-(p, \varepsilon b)$, $\mu \equiv p^{-\alpha}$ and $\nu \equiv |J(p, \varepsilon b)|$, so that

$$|C_n(f)| \leqslant \left[2L\|f\|_\infty p^\alpha + \|f\|_\infty^2 \frac{\Lambda_+(p, \varepsilon b)}{\Lambda_-(p, \varepsilon b}\right] \exp\left\{\frac{-\alpha \ln p\ \ln|J(p, \varepsilon b)|}{-\alpha \ln p + \ln|J(p, \varepsilon b)|} n\right\} \qquad \forall n \in \mathbb{Z}_+$$

and the estimated exponential rate is therefore (0.7), which completes the proof. □

Remark 1. Analogous estimates also hold in the non-mixing case $\varepsilon \in \mathbb{Q}$, but a further condition on the choice of the integer b is needed. More precisely, whenever $\varepsilon b/(p-1) \in \mathbb{Z}$ no exponential decay of correlations can be proved by using the methods previously exposed. In fact, it is easily seen that correlation decay may not even occur; this is, for instance, the case of the character

$$f(x, y) = \exp\left\{\mathrm{i}2\pi\left(-\frac{\varepsilon b}{p-1}x + by\right)\right\}$$

which obeys $(f \circ M)(x, y) = f(x, y)\,\mathrm{e}^{\mathrm{i}2\pi\omega b}$. Otherwise, $\varepsilon b/(p-1) \notin \mathbb{Z}$ intails an exponential decay of correlations. The proof of proposition 2 applies unchanged if $p\varepsilon b/(p-1) \notin \mathbb{Z}$ too, the only difference being that (2.10) holds for n large enough. The scale factor $((A/\mu) + B)$ must be obviously increased to cover any value of $n \in \mathbb{N}$, but the decay rate is exactly the same as in (0.7). The opposite case $p\varepsilon b/(p-1) \in \mathbb{Z}$ requires a slightly different investigation; the dominant term in the final upper bound to the correlations turns out to come from the piecewise approximation of g on Markov cylinders and the estimated rate is simply $p^{-\alpha}$.

Remark 2. If g is a piecewise constant function on Markov cylinders

$$g(x) = \sum_{h=0}^{p^{m^*}-1} c_h \mathcal{X}_{[h/p^{m^*},(h+1)/p^{m^*})}(x) \qquad c_h \in \mathbb{R}\ (\text{or}\ \mathbb{C}),\ \ m^* \in \mathbb{Z}_+ \tag{2.11}$$

relation (2.8) also holds, but without the term $2L\|f\|_\infty p^{-\alpha m}$ which would be derived from the approximation of g, which is unnecessary here. We simply have to put $m = m^*$ and consider $n > m^*$ to achieve an exponential bound to correlations with rate $|J(p, \varepsilon b)|$—whose extension to any $n \in \mathbb{N}$ is immediate. When $\varepsilon b/(p-1) \notin \mathbb{Z}$ but $p\varepsilon b/(p-1) \in \mathbb{Z}$, the disappearance of the approximation term leads to the stronger estimate

$$|C_n(f)| \leqslant C\,\mathrm{e}^{-\tau n^2} \qquad \forall n \in \mathbb{N}$$

with appropriate positive constants C and τ. It is noticeable that non-constant observables of the form $g(x)\,\mathrm{e}^{\mathrm{i}2\pi by}$, g given by (2.11), are discontinuous with respect to the metric d_2. As a conclusion, a wide family of discontinuous observables showing an (at least) exponential decay of correlations has been explicitly constructed.

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